Chase Joyner

802 Homework 4

April 3, 2017

Problem 11.2

As used in the proof to Theorem 11.2b, show that

$$\int_0^\infty t^a e^{-bt} dt = b^{-(a+1)}\Gamma(a+1).$$

Solution: Let $u = bt$. Then, by this change of variable we have

$$\int_0^\infty t^a e^{-bt} dt = \frac{1}{b}\int_0^\infty \left(\frac{u}{b}\right)^a e^{-u} du = b^{-(a+1)}\int_0^\infty u^{a+1-1} e^{-u} du = b^{-(a+1)}\Gamma(a+1),$$

where we obtain the last equality using the known Gamma function.

Problem 11.3

(a) Show that $(\mathbf{I} + \mathbf{XVX}')^{-1} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\mathbf{X}'$.

Solution: Notice that

$$\begin{aligned}
&(\mathbf{I} + \mathbf{XVX}') \times (\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\mathbf{X}') \\
&= \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\mathbf{X}' + \mathbf{XVX}' - \mathbf{XVX}'\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\mathbf{X}' \\
&= \mathbf{I} + \mathbf{XV}\left[\mathbf{V}^{-1}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1} + \mathbf{I} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\right]\mathbf{X}' \\
&= \mathbf{I} + \mathbf{XV}\left[\mathbf{I} - (\mathbf{V}^{-1} + \mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\right]\mathbf{X}' \\
&= \mathbf{I} + \mathbf{XV}[\mathbf{I} - \mathbf{I}]\mathbf{X}' = \mathbf{I}.
\end{aligned}$$

The exact argument above can also be used to show that

$$(\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X} + \mathbf{V}^{-1})^{-1}\mathbf{X}') \times (\mathbf{I} + \mathbf{XVX}') = \mathbf{I}.$$

This shows the result.

(b) Show that $(\mathbf{I}+\mathbf{X}\mathbf{V}\mathbf{X}')^{-1}\mathbf{X} = \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1}$.

Solution: Considering the difference and part (a), we find

$$\begin{aligned}
&(\mathbf{I}+\mathbf{X}\mathbf{V}\mathbf{X}')^{-1}\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} \\
&= \left[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{X}'\right]\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} \\
&= \mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{X}'\mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} \\
&= \mathbf{X} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1}) \\
&= \mathbf{X} - \mathbf{X} = \mathbf{0}
\end{aligned}$$

which shows the result.

(c) Show that $\mathbf{V}^{-1} - \mathbf{V}^{-1}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} = \mathbf{X}'(\mathbf{I}+\mathbf{X}\mathbf{V}\mathbf{X}')^{-1}\mathbf{X}$.

Solution: By equation (2.54) in the text, we obtain

$$\begin{aligned}
&\mathbf{V}^{-1} - \mathbf{V}^{-1}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} \\
&= \left[\mathbf{V} + (\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}'\mathbf{X})(\mathbf{X}'\mathbf{X})^{-1}\right]^{-1} \\
&= \left[(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{V}\right]^{-1}.
\end{aligned}$$

Now, by (2.54) again,

$$\left[(\mathbf{X}'\mathbf{X})^{-1} + \mathbf{V}\right]^{-1} = \mathbf{X}'\mathbf{X} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{X}'\mathbf{X}$$

and therefore we have that

$$\begin{aligned}
&\mathbf{V}^{-1} - \mathbf{V}^{-1}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{V}^{-1} \\
&= \mathbf{X}'\mathbf{X} - \mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{X}'\mathbf{X} \\
&= \mathbf{X}'\left[\mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X}+\mathbf{V}^{-1})^{-1}\mathbf{X}'\right]\mathbf{X} \\
&= \mathbf{X}'(\mathbf{I}+\mathbf{X}\mathbf{V}\mathbf{X}')^{-1}\mathbf{X}
\end{aligned}$$

where the last equality is from part (a).

Problem 5.20

(a) We can see that the t distribution is a mixture of normals using the following argument:

$$P(T_\nu \le t) = P\left(\frac{Z}{\sqrt{\chi^2_\nu/\nu}} \le t\right) = \int_0^\infty P(Z \le t\sqrt{x}/\sqrt{\nu})P(\chi^2_\nu = x)dx,$$

where T_ν is a t random variable with ν degrees of freedom. Using the Fundamental Theorem of Calculus and interpreting $P(\chi^2_\nu = x)$ as a pdf, we obtain

$$f_{T_\nu}(t) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2x/2\nu} \frac{\sqrt{x}}{\sqrt{\nu}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2-1}e^{-x/2}dx,$$

a scale mixture of normals. Verify this formula by direct integration.

Solution: Notice that

$$\begin{aligned}
&\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2x/2\nu} \frac{\sqrt{x}}{\sqrt{\nu}} \frac{1}{\Gamma(\nu/2)2^{\nu/2}} x^{\nu/2+1/2-1}e^{-x/2}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}\,\Gamma(\nu/2)2^{\nu/2}} \int_0^\infty x^{(\nu+1)/2-1} e^{-t^2x/2\nu - x/2}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}\,\Gamma(\nu/2)2^{\nu/2}} \int_0^\infty x^{(\nu+1)/2-1} e^{-x(t^2/2\nu+1/2)}dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\nu}\,\Gamma((\nu/2)2^{\nu/2}} \cdot \Gamma\big((\nu+1)/2\big) \frac{1}{(t^2/2\nu+1/2)^{(\nu+1)/2}} \\
&= \frac{1}{\sqrt{2\pi}} \frac{\Gamma\big((\nu+1)/2\big)}{\sqrt{\nu}\,\Gamma((\nu/2)2^{\nu/2}} \left(\frac{2\nu}{t^2+\nu}\right)^{(\nu+1)/2} \\
&= \frac{1}{\sqrt{\nu\pi}} \frac{\Gamma\big((\nu+1)/2\big)}{\Gamma((\nu/2)2^{\nu/2}} \left(\frac{1}{t^2+\nu}\right)^{(\nu+1)/2}
\end{aligned}$$

which is the pdf of a t_ν random variable. This shows the result.

(b) A similar formula holds for the F distribution; that is, it can be written as a mixture of chi squareds. If $F_{1,\nu}$ is an F random variable with 1 and ν degrees of freedom, then we can write

$$P(F_{1,\nu} \le \nu t) = \int_0^\infty P(\chi^2_1 \le ty) f_\nu(y)dy,$$

where $f_\nu(y)$ is a χ^2_ν pdf. Use the Fundamental Theorem of Calculus to obtain an integral expression for the pdf of $F_{1,\nu}$, and show that the integral equals the pdf.

Solution: Taking the derivative of both sides and interchanging the derivative with the integral, we have

$$
\begin{aligned}
\nu f_{1,\nu}(\nu t) &= \frac{d}{dt}\int_0^\infty P(\chi_1^2 \le ty) f_\nu(y)dy = \int_0^\infty \frac{d}{dt} P(\chi_1^2 \le ty) f_\nu(y)dy \\
&= \int_0^\infty y f_1(ty) f_\nu(y)dy \\
&= \int_0^\infty y \frac{1}{\sqrt{2}\,\Gamma(1/2)}(ty)^{-1/2}e^{-ty/2} \cdot \frac{1}{2^{\nu/2}\Gamma(\nu/2)} y^{\nu/2-1}e^{-y/2}dy \\
&= \frac{t^{-1/2}}{2^{(\nu+1)/2}\Gamma(1/2)\Gamma(\nu/2)} \int_0^\infty y^{(\nu+1)/2-1} e^{-(t+1)y/2} dy \\
&= \frac{t^{-1/2}}{2^{(\nu+1)/2}\Gamma(1/2)\Gamma(\nu/2)} \cdot \frac{\Gamma\big((\nu+1)/2\big)}{\big[(t+1)/2\big]^{(\nu+1)/2}} \\
&= \frac{t^{-1/2}}{\Gamma(1/2)\Gamma(\nu/2)} \cdot \frac{\Gamma\big((\nu+1)/2\big)}{(t+1)^{(\nu+1)/2}}.
\end{aligned}
$$

This implies that

$$
f_{1,\nu}(\nu t) = \frac{t^{-1/2}}{\nu\Gamma(1/2)\Gamma(\nu/2)} \cdot \frac{\Gamma\big((\nu+1)/2\big)}{(t+1)^{(\nu+1)/2}}.
$$

For clarity, consider $x = \nu t$ and so the above becomes

$$
\begin{aligned}
f_{1,\nu}(\nu t) &= \frac{\Gamma\big((\nu+1)/2\big)}{\nu\Gamma(1/2)\Gamma(\nu/2)} \frac{(x/\nu)^{-1/2}}{(1+x/\nu)^{(\nu+1)/2}} \\
&= \frac{\Gamma\big((\nu+1)/2\big)\nu^{\nu/2}}{\Gamma(1/2)\Gamma(\nu/2)} \frac{x^{1/2-1}}{(\nu+x)^{(1+\nu)/2}}
\end{aligned}
$$

and the RHS above is indeed the pdf of an $F_{1,\nu}$ random variable.